Harnack Inequality and Strong Feller Property for Stochastic Fast-Diffusion Equations *

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Abstract

As a continuation to [20], where the Harnack inequality and the strong Feller property are studied for a class of stochastic generalized porous media equations, this paper presents analogous results for stochastic fast-diffusion equations. Since the fast-diffusion equation possesses weaker dissipativity than the porous medium one does, some technical difficulties appear in the study. As a compensation to the weaker dissipativity condition, a Sobolev-Nash inequality is assumed for the underlying self-adjoint operator in applications. Some concrete examples are constructed to illustrate the main results.

1 Introduction

Recently, the dimension-free Harnack inequality introduced in [18] was established in [20] for a class of stochastic generalized porous media equations. As applications, the strong Feller property, estimates of the transition density and some contractivity properties were obtained for the associated Markov semigroup. The approach used in [20] is based on a coupling argument developed in [4], where Harnack inequalities are studied for diffusion semigroups on Riemannian manifolds with unbounded below curvatures. The advantage of this approach is that it avoids the assumption on curvature lower bounds used in previous articles (see [2, 3, 7, 15, 16]), which is very hard to verify in the framework of non-linear SPDEs. The main aim of this paper is to apply this method to stochastic generalized fast-diffusion equations studied in [14].

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Let $(E, \mathcal{M}, \mathbf{m})$ be a separable probability space and $(L, \mathcal{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ having discrete spectrum. Let

$$(0 <)\lambda_1 \le \lambda_2 \le \cdots$$

be all eigenvalues of $-L$ with unit eigenfunctions $\{e_i\}_{i\ge 1}$.

Next, let H be the completion of $L^2(\mathbf{m})$ under the inner product

$$\langle x, y\rangle_H := \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\langle x, e_i\rangle\langle y, e_i\rangle,$$

where and in what follows, $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbf{m})$. Let L_{HS} denote the space of all Hilbert-Schmidt operators from $L^2(\mathbf{m})$ to H. Let W_t be the cylindrical Brownian motion on $L^2(\mathbf{m})$ w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$; that is, $W_t = \{B_t^i e_i\}_{i\ge 1}$ for a sequence of independent one-dimensional $\mathcal{F}_t$-Brownian motions $\{B_t^i\}_{i\ge 1}$.

Let

$$\Psi : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$$

be progressively measurable and continuous in the second variable, and let

$$Q : [0,\infty) \times \Omega \to L_{HS}$$

be progressively measurable such that for all $t \ge 0$

$$\|Q_t\|_{L_{HS}}^2 \le q_t, \ a.s. \tag{1.1}$$

for q_t some deterministic local integrable function on $[0, +\infty)$. We consider the equation

$$\mathrm{d}X_t = \big\{L\Psi(t, X_t) + \gamma_t X_t\big\}\mathrm{d}t + Q_t \mathrm{d}W_t, \tag{1.2}$$

where $\gamma : [0,\infty) \to \mathbb{R}$ is locally bounded and measurable. In particular, if $\gamma = 0, Q = 0$ and $\Psi(t, s) = s^r := |s|^{r-1}s$ for some $r \in (0, 1)$, then (1.2) reduces back to the classical fast-diffusion equation (see e.g. [5]). For more general stochastic evolution equations in Hilbert space we refer to [8, 12, 13, 14] and the references within.

In general, for a fixed number $r \in (0, 1)$, we assume that there exist locally bounded positive measurable functions $\delta, \eta : [0,\infty) \to \mathbb{R}^+$ such that

$$\begin{aligned} &|\Psi(t, s)| \le \eta_t(1 + |s|^r), \quad s \in \mathbb{R}, t \ge 0,\\ &2\big(\Psi(t, s_1) - \Psi(t, s_2)\big)(s_1 - s_2) \ge \delta_t |s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}, \quad s_1, s_2 \in \mathbb{R}, t \ge 0, \end{aligned} \tag{1.3}$$

where δ satisfies $\inf\limits_{t\in[0,T]} \delta_t > 0$ for any $T > 0$. Due to the mean-valued theorem and the fact that $r < 1$, one has $(s_1 - s_2)(s_1^r - s_2^r) \ge r|s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}$. So, a simple example for (1.3) is that $\Psi(t, s) = \frac{\delta_t}{2r} s^r$ with $\eta_t = \frac{\delta_t}{2r}$.

According to [14, Theorem 3.9], for any $x \in H$ the equation (1.2) has a unique solution $X_t(x)$ with $X_0(x) = x$, which is a continuous adapted process on H satisfying

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t(x)\|_H^2 < \infty, \quad T > 0.$$

We intend to establish Harnack inequalities for

$$P_tF := \mathbb{E}F(X_t),\ F \in \mathfrak{M}_b(H),\ t > 0,$$

where $\mathfrak{M}_b(H)$ is the class of bounded measurable functions on H. Since the dissipativity condition (1.3) is essentially weaker than the corresponding one satisfied by the porous medium situation where $r > 1$, the method and results in [20] do not apply automatically in the present case.

As in [20], we assume that $Q_t(\omega)$ is non-degenerate for $t > 0$ and $\omega \in \Omega$; that is, $Q_t(\omega)x = 0$ implies $x = 0$. Let

$$\|x\|_{Q_t} := \begin{cases} \|y\|_2, & \text{if } y \in L^2(\mathbf{m}), Q_t y = x, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 1.1. *Assume (1.1) and (1.3) with $r \in (0,1)$. If there exist a constant $\sigma \geq \frac{4}{r+1}$ and a strictly positive function $\xi \in C([0,\infty))$ such that*

$$\|x\|_{r+1}^2 \cdot \|x\|_H^{\sigma-2} \geq \xi_t \|x\|_{Q_t}^\sigma, \quad x \in L^{r+1}(\mathbf{m}),\ t \geq 0 \tag{1.4}$$

holds on Ω, then for any $T > 0$, P_T is strong Feller and for any positive $F \in \mathfrak{M}_b(H)$, $p > 1$ and $x, y \in H$,

$$\begin{aligned} \frac{(P_TF)^p(y)}{P_TF^p(x)} \leq \exp\Big[&\frac{p-1}{4}\Big(2\int_0^T \theta_t \mathrm{d}t + \lambda_T T + \|x\|_H^2 + \|y\|_H^2\Big) \\ &+ (p-1)\frac{\int_0^T [(\sigma+2)g_t]^2 \mathrm{d}t}{4(\sigma\int_0^T g_t \mathrm{d}t)^2}\|x-y\|_H^2 \\ &+ \lambda_T^{\frac{2-\sigma}{2}}\Big(\frac{\sigma+2}{\sigma}\Big)^{\sigma+1}\frac{[2p(p+1)]^{\sigma/2}}{4(p-1)^{\sigma-1}\Big(\int_0^T g_t \mathrm{d}t\Big)^\sigma}\|x-y\|_H^\sigma\Big] \end{aligned} \tag{1.5}$$

holds for

$$\lambda_T := \frac{1}{2}\mathrm{e}^{-\int_0^T (2\gamma_t + 2q_t + 1)\mathrm{d}t} \inf_{t\in[0,T]} \delta_t,\ \theta_t := q_t + 2^{\frac{r+2}{r}} \eta_t^{\frac{r+1}{r}} \delta_t^{-\frac{1}{r}},\ g_t := (\delta_t \xi_t)^{\frac{1}{\sigma}} \mathrm{e}^{-\int_0^t \gamma_s \mathrm{d}s}.$$

Remark. (1) The right hand side of (1.5) comes from our argument and calculations, in particular the coupling method modified from [20] where the case $r \geq 1$ was studied. Comparing with known Gaussian type bounds in finite-dimensions, the first two terms in the exponential are natural. The third term of $\|x-y\|_H^\sigma$ for $\sigma \geq 4/(r+1)$ seems more technical, which appears when we handle the exponential moment of an additional term in the coupling by using the dissipasitivity of the drift $L\Psi$. It is not clear whether this term is exact or not, but since $\sigma > 2$ it does not destroy the short distance (or short time) behaviors of the heat kernel. On the other hand, due to the weaker dissipasitivity of the drift, it is reasonable for the semigroup to have worse long time behaviors.

(2) Harnack inequalities of type (1.5) has many applications. For instance, for diffusions on manifolds it has been applied to study the heat kernel estimate (cf. [10]), log-Sobolev

inequalities and contractivities of the semigroup (cf. [18, 1, 15]), and entropy-transportation inequalities (cf. [7]). In the symmetric case it was also applied to the study of transition probability kernels for infinite-dimensional diffusions (cf. [3, 2]). Here, due to the weaker dissipasitivity of the drift and the non-symmetry of the semigroup, we are not be able to derive stronger properties like hypercontractivity, which fails even in finite-dimensions e.g. the semigroup generated by $\Delta - \nabla|\cdot|^{r+1}$ on $\mathbb{R}^d$ for $r<1$, which is included in our model when E contains finite many elements. Thus, below we only present an application to moment estimates on heat kernels. To this end, we consider the following time-homogenous case.

Theorem 1.2. *Assume (1.1), (1.3) and that the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact. Let $\gamma \le 0$ be constant and Ψ, Q be deterministic and time-independent.*

(1) *The Markov semigroup P_t has a unique invariant probability measure μ and $\mu(e^{\varepsilon_0\|\cdot\|_H^{r+1}} + \|\cdot\|_{r+1}^{r+1}) < \infty$ for some $\varepsilon_0 > 0$. If $\gamma < 0$ then $\mu(e^{\varepsilon_0\|\cdot\|_H^2}) < \infty$ for some $\varepsilon_0 > 0$.*

(2) *If (1.4) holds for some constant $\xi > 0$, then μ has full support on H and for any $x \in H$, $T > 0$ and $p > 1$, the transition density $p_T(x,y)$ of P_T w.r.t μ satisfies*

$$\begin{aligned}\|p_T(x,\cdot)\|_{L^p(\mu)} \le \bigg\{\int_H \exp\Big[&-\frac{1}{4(p-1)}\Big(2\theta T + \lambda_T T + \|x\|_H^2 + \|y\|_H^2\Big) \\ &- \frac{\int_0^T [(\sigma+2)g_t]^2 \mathrm{d}t}{4(p-1)(\sigma\int_0^T g_t \mathrm{d}t)^2}\|x-y\|_H^2 \\ &- \lambda_T^{\frac{2-\sigma}{2}}\Big(\frac{\sigma+2}{\sigma}\Big)^{\sigma+1}\frac{2^{\frac{\sigma}{2}-2}[p(2p-1)]^{\sigma/2}}{(p-1)\Big(\int_0^T g_t\mathrm{d}t\Big)^{\sigma}}\|x-y\|_H^{\sigma}\Big]\mu(\mathrm{d}y)\bigg\}^{-(p-1)/p}.\end{aligned}$$

where

$$\lambda_T = \frac{\delta}{2}\mathrm{e}^{-(2\gamma+2q+1)T}, \quad \theta = q + 2^{\frac{r+2}{r}}\eta^{\frac{r+1}{r}}\delta^{-\frac{1}{r}}, \quad g_t = (\delta\xi)^{\frac{1}{\sigma}}\mathrm{e}^{-\gamma t}.$$

The above two theorems will be proved in the next section by modifying the argument in [20]. To apply Theorems 1.1 and 1.2, one has to verify (1.4) and the compactness of the embedding $L^{r+1}(\mathbf{m}) \subset H$. Since $r < 1$ so that the norm in $L^{r+1}(\mathbf{m})$, which is induced by the first drift term in (1.2), is normally incomparable with that in H, for (1.4) and the compactness of the embedding we shall need a Nash (or Sobolev) inequality. Along this line, explicit sufficient conditions for the main results to hold, as well as concrete examples, are presented in Section 3.

2 Proofs of Theorems 1.1 and 1.2

2.1 Proof of Theorem 1.1

As explained in [20], to prove the Harnack inequality for P_t, it suffices to construct a coupling (X_t, Y_t), which is a continuous adapted process on $H \times H$ such that

(i) X_t solves (1.2) with $X_0 = x$;
(ii) Y_t solves the equation

$$\mathrm{d}Y_t = \big\{L\Psi(t, Y_t) + \gamma_t Y_t\big\}\mathrm{d}t + Q_t \mathrm{d}\tilde{W}_t,\ Y_0 = y$$

for a cylindrical Brownian motion $\tilde{W}_t$ on $L^2(\mathbf{m})$ under a weighted probability measure $R\mathbb{P}$, where $\tilde{W}_t$ as well as the density R will be constructed later on by a Girsanov transformation;
(iii) $X_T = Y_T$, a.s.

As soon as (i)-(iii) are satisfied, then

$$\begin{aligned} P_T F(y) &= \mathbb{E} R F(Y_T) = \mathbb{E} R F(X_T) \\ &\le (\mathbb{E} R^{p/(p-1)})^{(p-1)/p} (\mathbb{E} F(X_T)^p)^{1/p} \\ &= (\mathbb{E} R^{p/(p-1)})^{(p-1)/p} (P_T F^p(x))^{1/p} \end{aligned} \tag{2.1}$$

which implies the desired Harnack inequality provided $\mathbb{E} R^{p/(p-1)} < \infty$.

To realize the above idea, for $\varepsilon > 0$ and $\beta \in \mathbf{C}([0,\infty); \mathbf{R}^+)$, let Y_t solve the equation

$$\mathrm{d}Y_t = \Big\{L\Psi(t, Y_t) + \gamma_t Y_t + \frac{\beta_t (X_t - Y_t)}{\|X_t - Y_t\|_H^\varepsilon} \mathbf{1}_{\{t<\tau\}}\Big\}\mathrm{d}t + Q_t \mathrm{d}W_t,\ Y_0 = y, \tag{2.2}$$

where $X_t := X_t(x)$ and $\tau := \inf\{t \ge 0 : X_t = Y_t\}$ is the coupling time.

According to [14, Theorem 3.9], we can prove that (2.2) also has a unique strong solution $Y_t(y)$ by using the same argument as in [20, Theorem A.2]. Hence, we have $X_t = Y_t$ for $t \ge \tau$ by the pathwise uniqueness of the solution.

Let

$$\zeta_t := \frac{\beta_t Q_t^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_H^\varepsilon} \mathbf{1}_{\{t<\tau\}}.$$

We have

$$\mathrm{d}Y_t = (L\Psi(t, Y_t) + \gamma_t Y_t)\mathrm{d}t + Q_t(\mathrm{d}W_t + \zeta_t \mathrm{d}t), \quad Y_0 = y.$$

According to the Girsanov theorem, $\tilde{W}_t := W_t + \int_0^t \zeta_s \mathrm{d}s$ is a cylindrical Brownian motion under $R\mathbb{P}$, where

$$R := \exp\Big[-\int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big]. \tag{2.3}$$

So, to verify (ii) and (iii), we need to find out $\varepsilon > 0$ and β such that

(a) $\tau \le T$ a.s.;
(b) $\mathbb{E}\exp\Big[\lambda \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big] < \infty, \quad \lambda > 0.$

By (1.3) and the Itô formula (see [14, Theorem A.2]),

$$\begin{aligned} \mathrm{d}\|X_t - Y_t\|_H^2 \le \big\{ &- \delta_t \mathbf{m}(|X_t - Y_t|^2 (|X_t| \vee |Y_t|)^{r-1}) \\ &+ 2\gamma_t \|X_t - Y_t\|_H^2 - 2\beta_t \|X_t - Y_t\|_H^{2-\varepsilon} \mathbf{1}_{\{t<\tau\}}\big\}\mathrm{d}t. \end{aligned} \tag{2.4}$$

This implies

$$
\begin{aligned}
(2.5)\qquad & \mathrm{d}\big\{\|X_t-Y_t\|_H^2 \mathrm{e}^{-2\int_0^t \gamma_s \mathrm{d}s}\big\} \\
& \le -\mathrm{e}^{-2\int_0^t \gamma_s \mathrm{d}s}\big\{\delta_t \mathbf{m}(|X_t-Y_t|^2(|X_t|\vee|Y_t|)^{r-1}) + 2\beta_t \|X_t-Y_t\|_H^{2-\varepsilon}\mathbf{1}_{\{t<\tau\}}\big\}\mathrm{d}t.
\end{aligned}
$$

Lemma 2.1. *If β satisfies $\int_0^T \beta_t \mathrm{e}^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\mathrm{d}t \ge \frac{1}{\varepsilon}\|x-y\|_H^\varepsilon$, then $X_T=Y_T$ a.s.*

Proof. By (2.4) and (2.5) we have

$$
\frac{2}{\varepsilon}\mathrm{d}\big\{\|X_t-Y_t\|_H^2 \mathrm{e}^{-2\int_0^t \gamma_s \mathrm{d}s}\big\}^{\varepsilon/2} \le -\beta_t \mathrm{e}^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\mathrm{d}t,\ \ t\le \tau\wedge T.
$$

If $T<\tau(\omega)$ for some ω, then

$$
\|X_T(\omega)-Y_T(\omega)\|_H^\varepsilon \mathrm{e}^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s} - \|x-y\|_H^\varepsilon \le -\varepsilon\int_0^T \beta_t \mathrm{e}^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\mathrm{d}t \le -\|x-y\|_H^\varepsilon.
$$

This implies $X_T(\omega)=Y_T(\omega)$, which is contradictory to $T<\tau(\omega)$. $\square$

From now on, we take $\varepsilon=\frac{\sigma}{\sigma+2}$ and

$$
\beta_t = c(\varepsilon\delta_t\xi_t)^{1/\sigma}\mathrm{e}^{-\frac{2}{\sigma+2}\int_0^t \gamma_s \mathrm{d}s},\quad c=\frac{\|x-y\|_H^\varepsilon}{\varepsilon\int_0^T(\varepsilon\delta_t\xi_t)^{\frac{1}{\sigma}}\mathrm{e}^{-\int_0^t \gamma_s \mathrm{d}s}\mathrm{d}t},
$$

so that (a) holds according to Lemma 2.1. Let $f_t := \Big(\mathbf{m}[(|X_t|\vee|Y_t|)^{r+1}]\Big)^{\frac{1-r}{1+r}}$. By (2.5), the Hölder inequality and (1.4) we obtain

$$
\begin{aligned}
\mathrm{d}\big\{\|X_t-Y_t\|_H^2 \mathrm{e}^{-2\int_0^t \gamma_s \mathrm{d}s}\big\}^\varepsilon &\le -\varepsilon\delta_t \mathrm{e}^{-2\varepsilon\int_0^t \gamma_s \mathrm{d}s}\|X_t-Y_t\|_H^{2(\varepsilon-1)}\mathbf{m}(|X_t-Y_t|^2(|X_t|\vee|Y_t|)^{r-1})\mathrm{d}t \\
&\le -\varepsilon\delta_t \mathrm{e}^{-2\varepsilon\int_0^t \gamma_s \mathrm{d}s}\|X_t-Y_t\|_H^{2(\varepsilon-1)}\frac{\|X_t-Y_t\|_{r+1}^2}{\Big(\mathbf{m}[(|X_t|\vee|Y_t|)^{r+1}]\Big)^{\frac{1-r}{1+r}}}\mathrm{d}t \\
&\le -\varepsilon\delta_t\xi_t \mathrm{e}^{-2\varepsilon\int_0^t \gamma_s \mathrm{d}s}\frac{\|X_t-Y_t\|_{Q_t}^\sigma}{\|X_t-Y_t\|_H^{\sigma-2\varepsilon}f_t}\mathrm{d}t \\
&= -\varepsilon\delta_t\xi_t \mathrm{e}^{-2\varepsilon\int_0^t \gamma_s \mathrm{d}s}\frac{\|X_t-Y_t\|_{Q_t}^\sigma}{\|X_t-Y_t\|_H^{\sigma\varepsilon}f_t}\mathrm{d}t \\
&= -\frac{\beta_t^\sigma\|X_t-Y_t\|_{Q_t}^\sigma}{c^\sigma\|X_t-Y_t\|_H^{\sigma\varepsilon}f_t}\mathrm{d}t.
\end{aligned}
$$

Combining this with the Hölder inequality and the fact that

$$
\sup_{a>0}\{a^{\frac{\sigma-2}{\sigma}}b^{\frac{2}{\sigma}}-a\} = \Big[\Big(\frac{\sigma-2}{\sigma}\Big)^{\frac{\sigma-2}{2}} - \Big(\frac{\sigma-2}{\sigma}\Big)^{\frac{\sigma}{2}}\Big]b \le b,\quad b>0
$$

implies $a^{\frac{\sigma-2}{\sigma}} b^{\frac{2}{\sigma}} \le a+b,\ a,b>0$, we arrive at

$$
\begin{aligned}
\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t &= \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_{Q_t}^2}{\|X_t - Y_t\|_H^{2\varepsilon}} \mathrm{d}t \\
&\le \Big(\int_0^T f_t^{\frac{2}{\sigma-2}} \mathrm{d}t \Big)^{\frac{\sigma-2}{\sigma}} \Big(\int_0^T \frac{\beta_t^\sigma \|X_t - Y_t\|_{Q_t}^\sigma}{\|X_t - Y_t\|_H^{\sigma\varepsilon} f_t} \mathrm{d}t \Big)^{\frac{2}{\sigma}} \\
&\le \Big(\int_0^T f_t^{\frac{2}{\sigma-2}} \mathrm{d}t \Big)^{\frac{\sigma-2}{\sigma}} \Big(c^\sigma \|x-y\|_H^{2\varepsilon} \Big)^{\frac{2}{\sigma}} \\
&\le \lambda \int_0^T f_t^{\frac{2}{\sigma-2}} \mathrm{d}t + \lambda^{(2-\sigma)/2} c^\sigma \|x-y\|_H^{2\varepsilon}, \quad \lambda > 0.
\end{aligned} \tag{2.6}
$$

Since $\sigma \ge \frac{4}{1+r}$ implies $\frac{2}{\sigma-2} \le \frac{1+r}{1-r}$, we have

$$
f_t^{\frac{2}{\sigma-2}} \le \mathbf{m}\big(1 + |X_t|^{r+1} \vee |Y_t|^{r+1}\big)^{\frac{2(1-r)}{(\sigma-2)(1+r)}} \le \mathbf{m}\big(1 + |X_t|^{r+1} \vee |Y_t|^{r+1}\big).
$$

Thus,

$$
\begin{aligned}
&\mathbb{E} \exp\Big[\lambda \int_0^T f_t^{\frac{2}{\sigma-2}} \mathrm{d}t\Big] \\
&\le \mathbb{E} \exp\Big[\lambda \int_0^T (1 + \|X_t\|_{r+1}^{r+1} + \|Y_t\|_{r+1}^{r+1}) \mathrm{d}t\Big], \quad \lambda > 0.
\end{aligned} \tag{2.7}
$$

Therefore, to verify (b) we need to prove that $\int_0^T (\|X_t\|_{r+1}^{r+1} + \|Y_t\|_{r+1}^{r+1}) \mathrm{d}t$ is exponentially integrable. This follows from the following Lemma.

Lemma 2.2. *We have*

$$
\mathbb{E} \exp\Big[\lambda_T \int_0^T \|X_t\|_{r+1}^{r+1} \mathrm{d}t\Big] \le \exp\Big[\int_0^T \theta_t \mathrm{d}t + \|x\|_H^2\Big], \tag{2.8}
$$

$$
\begin{aligned}
&\mathbb{E} \exp\Big[\lambda_T \int_0^T \|Y_t\|_{r+1}^{r+1} \mathrm{d}t\Big] \\
&\le \exp\Big[\int_0^T \theta_t \mathrm{d}t + \|y\|_H^2 + \|x-y\|_H^{2(1-\varepsilon)} \int_0^T \beta_t^2 e^{-2\varepsilon \int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t\Big],
\end{aligned} \tag{2.9}
$$

where $\lambda_T = \frac{1}{2} \exp[-\int_0^T (2\gamma_s + 2q_s + 1) \mathrm{d}s] \inf_{t \in [0,T]} \delta_t$.

Proof. Since assumption (1.3) implies

$$
\begin{aligned}
-2\langle \Psi(t, X_t), X_t \rangle &= -2\langle \Psi(t, X_t) - \Psi(t, 0), X_t - 0 \rangle - 2\Psi(t,0)\mathbf{m}(X_t) \\
&\le -\delta_t \|X_t\|_{r+1}^{r+1} + 2\eta_t \|X_t\|_{r+1},
\end{aligned}
$$

by the Itô formula we obtain

$$
\begin{aligned}
\mathrm{d}\|X_t\|_H^2 &\le \big\{ -\delta_t \|X_t\|_{r+1}^{r+1} + 2\eta_t \|X_t\|_{r+1} + 2\gamma_t \|X_t\|_H^2 + q_t \big\} \mathrm{d}t + 2\langle X_t, Q_t \mathrm{d}W_t \rangle \\
&\le \Big\{ \theta_t - \frac{\delta_t}{2} \|X_t\|_{r+1}^{r+1} + 2\gamma_t \|X_t\|_H^2 \Big\} \mathrm{d}t + 2\langle X_t, Q_t \mathrm{d}W_t \rangle.
\end{aligned} \tag{2.10}
$$

Recall that $\theta_t = q_t + 2^{\frac{r+2}{r}}\eta_t^{\frac{r+1}{r}}\delta_t^{-\frac{1}{r}}$ and $q_t \ge \sup_{\omega\in\Omega}\|Q_t\|^2_{L_{HS}}$. This implies

$$\begin{aligned}
&\mathrm{d}\Big\{\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\|X_t\|_H^2\Big\}\\
&\le \mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\Big\{\theta_t - \frac{\delta_t}{2}\|X_t\|_{r+1}^{r+1} - 2q_t\|X_t\|_H^2\Big\}\mathrm{d}t + 2\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\langle X_t, Q_t\mathrm{d}W_t\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\delta_{0,T}}{2}\mathrm{e}^{-\int_0^T(2\gamma_s+2q_s)\mathrm{d}s}\int_0^T\|X_t\|_{r+1}^{r+1}\mathrm{d}t\\
&\le \int_0^T\theta_t\mathrm{d}t + \|x\|_H^2 + M_T - \int_0^T 2q_t\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\|X_t\|_H^2\mathrm{d}t
\end{aligned}$$

where $\delta_{0,T} := \inf_{t\in[0,T]}\delta_t$ and $M_T = 2\int_0^T \mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\langle X_t, Q_t\mathrm{d}W_t\rangle$. It is easy to check from (2.10) and (1.1) that M_t is a martingale. Taking $\lambda_T = \frac{\delta_{0,T}}{2}\mathrm{e}^{-\int_0^T(2\gamma_s+2q_s+1)\mathrm{d}s}$, we obtain

$$\begin{aligned}
&\mathbb{E}\exp\Big[\lambda_T\int_0^T\|X_t\|_{r+1}^{r+1}\mathrm{d}t\Big]\\
&\le \exp\Big[\int_0^T\theta_t\mathrm{d}t + \|x\|_H^2\Big]\mathbb{E}\exp\Big[M_T - \int_0^T 2q_t\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\|X_t\|_H^2\mathrm{d}t\Big].
\end{aligned}$$

Since $\langle M\rangle_t \le \int_0^T 4q_t\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\|X_t\|_H^2\mathrm{d}t$ and $\mathbb{E}\exp[M_t - \frac{1}{2}\langle M\rangle_t] = 1$, we obtain

$$\mathbb{E}\exp\Big[M_T - \int_0^T 2q_t\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s)\mathrm{d}s}\|X_t\|_H^2\mathrm{d}t\Big] \le 1.$$

Thus, (2.8) holds.

Similarly, since (2.4) implies $\|X_t - Y_t\|_H^2 \le \mathrm{e}^{2\int_0^t\gamma_s\mathrm{d}s}\|x-y\|_H^2$, by (2.2) and the Itô formula we have

$$\begin{aligned}
&\mathrm{d}\Big\{\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s+1)\mathrm{d}s}\|Y_t\|_H^2\Big\}\\
\le\ & \mathrm{e}^{-\int_0^t(2\gamma_s+2q_s+1)\mathrm{d}s}\Big[\theta_t - \frac{\delta_t}{2}\|Y_t\|_{r+1}^{r+1} - (2q_t+1)\|Y_t\|_H^2\\
&+2\|Y_t\|_H\beta_t\|X_t-Y_t\|_H^{1-\varepsilon}\mathbf{1}_{\{t<\tau\}}\Big]\mathrm{d}t + \mathrm{d}M_t'\\
\le\ & \mathrm{e}^{-\int_0^t(2\gamma_s+2q_s+1)\mathrm{d}s}\Big[\theta_t - \frac{\delta_t}{2}\|Y_t\|_{r+1}^{r+1} - 2q_t\|Y_t\|_H^2 + \beta_t^2\|X_t-Y_t\|_H^{2(1-\varepsilon)}\mathbf{1}_{\{t<\tau\}}\Big]\mathrm{d}t + \mathrm{d}M_t'\\
\le\ & \mathrm{e}^{-\int_0^t(2\gamma_s+2q_s+1)\mathrm{d}s}\Big[\theta_t - \frac{\delta_t}{2}\|Y_t\|_{r+1}^{r+1} - 2q_t\|Y_t\|_H^2 + \beta_t^2\mathrm{e}^{2(1-\varepsilon)\int_0^t\gamma_s\mathrm{d}s}\|x-y\|_H^{2(1-\varepsilon)}\Big]\mathrm{d}t + \mathrm{d}M_t',
\end{aligned}$$

where $M_t' := \int_0^t 2\mathrm{e}^{-\int_0^s(2\gamma_u+2q_u+1)\mathrm{d}u}\langle Y_s, B_s\mathrm{d}W_s\rangle$ is a martingale. This implies

$$\begin{aligned}
&\frac{\delta_{0,T}}{2}\mathrm{e}^{-\int_0^T(2\gamma_s+2q_s+1)\mathrm{d}s}\int_0^T\|Y_t\|_{r+1}^{r+1}\mathrm{d}t\\
&\le \int_0^T\theta_t\mathrm{d}t + \|y\|_H^2 + \|x-y\|_H^{2(1-\varepsilon)}\int_0^T\beta_t^2\mathrm{e}^{-2\varepsilon\int_0^t\gamma_s\mathrm{d}s}\mathrm{d}t\\
&\quad + M_T' - \int_0^T 2q_t\mathrm{e}^{-\int_0^t(2\gamma_s+2q_s+1)\mathrm{d}s}\|Y_t\|_H^2\mathrm{d}t.
\end{aligned}$$

Therefore, taking $\lambda_T = \frac{\delta_{0,T}}{2} \mathrm{e}^{-\int_0^T (2\gamma_s + 2q_s + 1)\mathrm{d}s}$ and noting that

$$\langle M' \rangle_T \le \int_0^T 4 q_t \mathrm{e}^{-\int_0^t (2\gamma_s + 2q_s + 1)\mathrm{d}s} \|Y_t\|_H^2 \mathrm{d}t,$$

we obtain (2.9). $\square$

Now, combining (2.1) and (2.3) we obtain

$$\begin{aligned}
\frac{(P_T F(y))^p}{P_T F^p(x)} &\le \Big(\mathbb{E} R^{p/(p-1)}\Big)^{p-1} \\
&= \Big\{\mathbb{E} \exp\Big[\frac{p}{p-1} \int_0^T \langle \zeta_t, \mathrm{d}W_t \rangle - \frac{p}{2(p-1)} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big]\Big\}^{p-1} \\
&\le \Big\{\mathbb{E} \exp\Big[\frac{qp}{p-1} \int_0^T \langle \zeta_t, \mathrm{d}W_t \rangle - \frac{q^2 p^2}{2(p-1)^2} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big]\Big\}^{\frac{p-1}{q}} \\
&\quad \cdot \Big\{\mathbb{E} \exp\Big[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big]\Big\}^{\frac{(q-1)(p-1)}{q}} \\
&= \Big\{\mathbb{E} \exp\Big[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big]\Big\}^{\frac{(q-1)(p-1)}{q}}, \quad q > 1.
\end{aligned} \tag{2.11}$$

Moreover, letting $\lambda = \frac{\lambda_T (q-1)(p-1)^2}{pq(pq-p+1)}$, by (2.6), (2.7) and Lemma 2.2 we obtain that

$$\begin{aligned}
&\mathbb{E} \exp\Big[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big] \\
&\le \mathbb{E} \exp\Big[\frac{\lambda_T}{2} \int_0^T (1 + \|X_t\|_{r+1}^{r+1} + \|Y_t\|_{r+1}^{r+1}) \mathrm{d}t \\
&+ \frac{qp(qp-p+1)}{2(q-1)(p-1)^2} \Big(\frac{\lambda_T (q-1)(p-1)^2}{pq(pq-p+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_H^{2\varepsilon}\Big] \\
&\le \exp\Big[\frac{1}{2}\Big(2\int_0^T \theta_t \mathrm{d}t + \lambda_T T + \|x\|_H^2 + \|y\|_H^2 + \|x-y\|_H^{2(1-\varepsilon)} \int_0^T \beta_t^2 e^{-2\varepsilon \int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t\Big) \\
&\qquad + \frac{qp(qp-p+1)}{2(q-1)(p-1)^2} \Big(\frac{\lambda_T (q-1)(p-1)^2}{pq(pq-p+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_H^{2\varepsilon}\Big].
\end{aligned} \tag{2.12}$$

Combing this with (2.11) and simply letting $q = 2$, we obtain

$$\begin{aligned}
\frac{(P_T F(y))^p}{P_T F^p(x)} &\le \exp\Big[\frac{p-1}{4}\Big(2\int_0^T \theta_t \mathrm{d}t + \lambda_T T + \|x\|_H^2 + \|y\|_H^2 + \\
&\|x-y\|_H^{2(1-\varepsilon)} \int_0^T \beta_t^2 e^{-2\varepsilon \int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t\Big) + \frac{p(p+1)}{2(p-1)} \Big(\frac{\lambda_T (p-1)^2}{2(p+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_H^{2\varepsilon}\Big].
\end{aligned} \tag{2.13}$$

Then the desired result (1.5) follows by using the definition of β_t and c.

Finally, since

$$|P_T F(y) - P_T F(x)| = |\mathbb{E}(R-1)F(X_T)| \le \|F\|_\infty \mathbb{E}|R-1|,$$

and since due to (2.11) R is uniformly integrable for $\|x-y\|_H \le 1$, by the dominated convergence theorem we obtain

$$\lim_{y\to x} |P_T F(y) - P_T F(x)| \le \|F\|_\infty \lim_{y\to x} \mathbb{E}|R-1| = \|F\|_\infty \mathbb{E} \lim_{y\to x} |R-1| = 0$$

for any bounded measurable function F on H, where the last equality follows from $\lim_{y\to x} R = 1$ due to (2.6). So, P_T is strong Feller. Now the proof is complete. □

2.2 Proof of Theorem 1.2

Since the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact, the existence and uniqueness of the invariant measure μ follow from b) of §4 in [8] for $\mathbf{m}(|\cdot|^{r+1})$ in place of $\mathbf{m}(|\cdot|^2)$. So, for (1) it suffices to prove the desired concentration property. By (1.3) we have

$$\begin{aligned} \mathrm{d}\|X_t\|_H^2 &\le (c - \theta\|X_t\|_{r+1}^{r+1} + 2\gamma\|X_t\|_H^2)\mathrm{d}t + 2\langle X_t, Q\mathrm{d}W_t\rangle \\ &\le (c - \theta\|X_t\|_{r+1}^{r+1})\mathrm{d}t + 2\langle X_t, Q\mathrm{d}W_t\rangle, \end{aligned}$$

where $c, \theta > 0$ are two constants. Then, by a standard argument as in [20] we obtain

$$\mu(\|\cdot\|_{r+1}^{r+1}) < \infty.$$

If $\gamma < 0$ and ε_0 is small enough, then the Itô formula implies

$$\begin{aligned} \mathrm{d}e^{\varepsilon_0\|X_t\|_H^2} &\le \Big(c - \theta\|X_t\|_{r+1}^{r+1} + 2\gamma\|X_t\|_H^2 + \frac{\varepsilon_0}{2} q\|X_t\|_H^2\Big)\varepsilon_0 e^{\varepsilon_0\|X_t\|_H^2}\mathrm{d}t + \mathrm{d}M_t \\ &\le (c_1 - \theta_1 e^{\varepsilon_0\|X_t\|_H^2})\mathrm{d}t + \mathrm{d}M_t \end{aligned}$$

for some constants $c_1, \theta_1 > 0$ and local martingale $M_t := 2\varepsilon_0 \int_0^t e^{\varepsilon_0\|X_s\|_H^2}\langle X_s, Q\mathrm{d}W_s\rangle$. This implies

$$\mu_n(e^{\varepsilon_0\|\cdot\|_H^2}) = \frac{1}{n}\int_0^n \mathbb{E}e^{\varepsilon_0\|X_t(0)\|_H^2}\mathrm{d}t \le \frac{c_1}{\theta_1}$$

for $\mu_n := \frac{1}{n}\int_0^n (\delta_0 P_t)\mathrm{d}t$. Since μ is the weak limit of a subsequence of μ_n (see the proof of [14, Proposition 2.2]), we obtain $\mu(e^{\varepsilon_0\|\cdot\|_H^2}) < \infty$. Similarly, $\mu(e^{\varepsilon_0\|\cdot\|_H^{r+1}}) < \infty$ holds for $\gamma = 0$.

Finally, the full support of μ and the L^p-estimate of the transition density follows from the Harnack inequality (1.5) by repeating the proof of Theorem 1.2 (1) and (2) in [20]. □

3 Explicit sufficient conditions and Examples

To provide explicit sufficient conditions for (1.4), we need the following Nash inequality:

$$\|f\|_2^{2+4/d} \le C\langle f, -Lf\rangle,\ f \in \mathcal{D}(L),\ \mathbf{m}(|f|) = 1. \tag{3.1}$$

Lemma 3.1. *Let $r \in (0,1)$. Assume that (3.1) holds for some $d \in (0, \frac{2(r+1)}{1-r})$ and $-(-L)^{1/n}$ is a Dirichlet operator for some $n \geq 1$. Then the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact. In particular,*

$$\|x\|_H := \langle x, (-L)^{-1}x\rangle^{1/2} \leq c\|x\|_{r+1},\ x \in L^{r+1}(\mathbf{m})$$

holds for some $c > 0$.

Proof. Take $\varepsilon \in (0,1)$ such that $d_\varepsilon := d/\varepsilon \in (d, \frac{2(r+1)}{1-r})$, and let $L_\varepsilon := -(-L)^\varepsilon$. By [6, Theorem 1.3] and (3.1),

$$\|f\|_2^{2+4/d_\varepsilon} \leq C'\langle f, -L_\varepsilon f\rangle,\ f \in \mathcal{D}(L_\varepsilon),\ \mathbf{m}(|f|) = 1$$

holds for some constant $C' > 0$. By this and again [6, Theorem 1.3] and (3.1), we have

$$\|f\|_2^{2+\frac{4}{d_\varepsilon n}} \leq c_0\langle f, (-L_\varepsilon)^{1/n} f\rangle, \quad f \in \mathcal{D}((-L_\varepsilon)^{1/n}),\ \mathbf{m}(|f|) = 1 \tag{3.2}$$

for some $c_0 > 0$. Let T_t be the semigroup generated by $-(-L_\varepsilon)^{1/n}$, which is sub-Markovian since $-(-L_\varepsilon)^{1/n} = -(-L)^{\varepsilon/n}$ is a Dirichlet operator. Then it follows from (3.2) that (see [9])

$$\|T_t\|_{1\to\infty} \leq c_1 t^{-d_\varepsilon n/2}, \quad t > 0$$

holds for some constant $c_1 > 0$. Since T_t is contractive in $L^1(\mathbf{m})$ and symmetric in $L^2(\mathbf{m})$, by this and the Riesz-Thorin interpolation theorem we obtain

$$\|T_t\|_{1\to 2} = \|T_t\|_{2\to\infty} \leq c_2 t^{-d_\varepsilon n/4}, \quad t > 0 \tag{3.3}$$

for some constant $c_2 > 0$. Moreover, since $\lambda_1 > 0$ so that $\|T_t\|_{2\to 2} \leq \mathrm{e}^{-\lambda_1 t}$ for $t > 0$, (3.3) yields

$$\|T_t\|_{1\to\infty} \leq \|T_{t/4}\|_{1\to 2}\|T_{t/2}\|_{2\to 2}\|T_{t/4}\|_{2\to\infty} \leq c_3 t^{-d_\varepsilon n/2} e^{-\lambda_1^{\frac{\varepsilon}{n}} t/2}, \quad t > 0$$

for some $c_3 > 0$. By this and the Riesz-Thorin interpolation theorem we conclude that for any $1 < p < q$,

$$\|T_t\|_{p\to q} \leq \|T_t\|_{1\to\infty}^{\frac{q-p}{pq}} \leq c_4[t^{-d_\varepsilon n/2} e^{-\lambda_1^{\frac{\varepsilon}{n}} t/2}]^{\frac{q-p}{pq}}, \quad t > 0$$

holds for some constant $c_4 > 0$. Therefore,

$$C_{p,q} := \int_0^\infty \|T_t\|_{p\to q} dt < \infty$$

provided $\frac{q-p}{pq} < \frac{2}{d_\varepsilon n}$. Thus,

$$\|(-L_\varepsilon)^{-1/n}\|_{p\to q} \leq C_{p,q} < \infty, \quad \frac{q-p}{pq} < \frac{2}{d_\varepsilon n}.$$

Since $d_\varepsilon < \frac{2(r+1)}{1-r}$, letting $p_i := \frac{r+1}{1-2(i-1)(r+1)/d_\varepsilon n}$ $(1 \leq i \leq n+1)$, one has

$$p_1 = r+1, \quad \frac{p_{i+1} - p_i}{p_{i+1}p_i} = \frac{2}{d_\varepsilon n}\ (1 \leq i \leq n) \ \text{ and } \ p_{n+1} = \frac{r+1}{1-2(r+1)/d_\varepsilon} > \frac{r+1}{r}.$$

So, there exist $r+1 =: p'_1 < p'_2 < \cdots < p'_{n+1} := \frac{r+1}{r}$ such that $\frac{p'_{i+1}-p'_i}{p'_{i+1}p'_i} < \frac{2}{d_\varepsilon n},\ 1 \le i \le n$. Therefore,

$$c^2 := \|(-L_\varepsilon)^{-1}\|_{r+1\to(r+1)/r} \le \prod_{i=1}^n \|(-L_\varepsilon)^{-\frac{1}{n}}\|_{p'_i\to p'_{i+1}} \le \prod_{i=1}^n C_{p'_i,p'_{i+1}} < \infty.$$

This implies

$$\begin{aligned}&\langle x, (-L_\varepsilon)^{-1}x\rangle \le \|x\|_{r+1}\|(-L_\varepsilon)^{-1}x\|_{(r+1)/r}\\ &\le \|x\|_{r+1}^2\|(-L_\varepsilon)^{-1}\|_{r+1\to(r+1)/r} = c^2\|x\|_{r+1}^2,\quad x\in L^{r+1}(\mathbf{m}).\end{aligned}$$

Then the proof is completed since $\{x \in L^2(\mathbf{m}):\ \langle x, (-L_\varepsilon)^{-1}x\rangle \le N\}$ is relatively compact in H for any $N > 0$. □

Corollary 3.2. *Let $Q_t e_i = q_i e_i$ for $i \ge 1$ with $\sum_{i=1}^\infty \frac{q_i^2}{\lambda_i} < \infty$, so that Q is Hilbert-Schmidt from $L^2(\mathbf{m})$ to H. If $\varepsilon \in (0,1)$ and L satisfies (3.1) for some $d \in (0, \frac{2\varepsilon(1+r)}{1-r})$, $-(-L)^{1/n}$ is a Dirichlet operator for some $n \ge 1$ and there exist $c > 0$, $\sigma \ge \frac{4}{1+r}$ such that*

$$q_i \ge c\lambda_i^{\frac{\sigma+2\varepsilon-2}{2\sigma}},\ i\ge 1.$$

Then the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact and (1.4) holds for the same σ.

Proof. By Lemma 3.1, it suffices to verify (1.4). By the Hölder inequality,

$$\begin{aligned}\|x\|_Q^\sigma &= \Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 q_i^{-2}\Big)^{\sigma/2} = \Big(\sum_{i=1}^\infty \frac{\langle x, e_i\rangle^2}{\lambda_i^{\frac{\sigma-2}{\sigma}}}\lambda_i^{\frac{\sigma-2}{\sigma}} q_i^{-2}\Big)^{\sigma/2}\\ &\le \Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 \lambda_i^{\frac{\sigma-2}{2}} q_i^{-\sigma}\Big)\Big(\sum_{i=1}^\infty \frac{\langle x, e_i\rangle^2}{\lambda_i}\Big)^{\frac{\sigma-2}{2}}\\ &= \|x\|_H^{\sigma-2}\Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2\lambda_i^{\frac{\sigma-2}{2}}q_i^{-\sigma}\Big)\\ &\le c^{-\sigma}\|x\|_H^{\sigma-2}\Big(\sum_{i=1}^\infty\langle x, e_i\rangle^2\lambda_i^{-\varepsilon}\Big).\end{aligned}\tag{3.4}$$

By (3.1) and [6, Theorem 1.3], there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_2^{2+4\varepsilon/d} \le C_\varepsilon\langle f, (-L)^\varepsilon f\rangle,\ f\in\mathcal{D}((-L)^\varepsilon),\ \mathbf{m}(|f|) = 1.$$

Applying Lemma 3.1 to $-(-L)^\varepsilon$ in place of L, there exists a constant $c_1 > 0$ such that

$$\|x\|_{r+1}^2 \ge c_1\|(-L)^{-\varepsilon/2}x\|_2^2 = c_1\sum_{i=1}^\infty \langle x, e_i\rangle^2\lambda_i^{-\varepsilon}.$$

Combining this with (3.4), we obtain (1.4) for some constant $\xi > 0$.

□

Example 3.3. *Let $Q_te_i = q_ie_i (i \geq 1)$, $\Psi(t,x) := |x|^{r-1}x$ and $\gamma_t := c$ for some constant $c < 0$. Let $L := \Delta$ be the Laplace operator on a bounded domain in $\mathbf{R}$ with Dirichlet boundary conditions. If $r \in (\frac{1}{3}, 1)$, then all assertions in Theorem 1.1 and 1.2 hold provided there exist constants $c_1, c_2 > 0$, $\alpha < 1$ and $\varepsilon \in (\frac{1-r}{2(1+r)}, \frac{r}{1+r})$ such that*

$$c_1 i^{\varepsilon(r+1)+1-r} \leq q_i^2 \leq c_2 i^\alpha, \ i \geq 1.$$

Proof. Since $\lambda_i \geq ci^2$ for some constant $c > 0$, by Corollary 3.2 the above conditions imply (1.4). □

Remark : We can also consider the case where L is the Laplace operator on a post critical finite self-similar fractal with $s > 0$ the Hausdorff dimension of the fractal in the effective resistance metric. In this case we has $\lambda_i \geq ci^{(s+1)/s}, \ i \geq 1$ for some $c > 0$ according to [11, Theorem 2.11].

To construct examples for our results on high dimensional spaces, we may e.g. take $L = -(-\Delta)^\alpha$ for large enough $\alpha > 0$. More generally, let $-L_0$ be a self-adjoint Dirichlet operator on $L^2(\mathbf{m})$ with discrete spectrum

$$(0 <)\lambda_1^{(0)} \leq \lambda_2^{(0)} \leq \cdots$$

and the corresponding unit eigenfunctions $\{e_i\}_{i\geq 1}$ be an ONB on $L^2(\mathbf{m})$. As in Corollary 3.2, let $Qe_i := q_ie_i$ for a sequence $\{q_i \neq 0\}_{i\geq 1}$. Let, for simplicity, $\gamma_t = -c_0$ and $\Psi \in C(\mathbb{R})$ satisfy

$$\begin{aligned} &|\Psi(s)| \leq \eta(1 + |s|^r), \quad s \in \mathbb{R}, \ t \geq 0, \\ &2\langle \Psi(s_1) - \Psi(s_2), s_1 - s_2\rangle \geq \delta\mathbf{m}(|s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}), \quad s_1, s_2 \in \mathbb{R}, \ t \geq 0 \end{aligned} \tag{3.5}$$

for some $c_0 \geq 0$ and $\eta, \delta > 0$. For any positive constant α, we consider the equation (1.2) for

$$L := -(-L_0)^\alpha = -\sum_{i=1}^\infty (\lambda_i^{(0)})^\alpha \langle e_i, \cdot\rangle e_i.$$

That is, consider

$$\mathrm{d}X_t = -\big\{(-L_0)^\alpha \Psi(X_t) + c_0X_t\big\}\mathrm{d}t + Q\mathrm{d}W_t. \tag{3.6}$$

Proposition 3.4. *Let L_0 satisfy (3.1) with $d \in (0, \frac{2\varepsilon(1+r)}{1-r})$ for some $\varepsilon \in (0,1)$, and Ψ satisfy (3.5). If there exists $\alpha > \frac{d(1-r)}{2\varepsilon(1+r)}$ such that*

$$\sum_{i=1}^\infty \frac{q_i^2}{(\lambda_i^{(0)})^\alpha} < \infty, \quad q_i \geq c(\lambda_i^{(0)})^{\frac{\alpha(\sigma+2\varepsilon-2)}{2\sigma}}, \ i \geq 1, \tag{3.7}$$

where $\sigma \geq \frac{4}{1+r}$ is a constant, then the Markov semigroup of the solution to (3.6) satisfies all assertions in Theorems 1.1 and 1.2 for the same σ and some $\xi > 0$.

Proof. We only need to notice that the eigenvalues of $L := -(-L_0)^\alpha$ are

$$-\lambda_i := -(\lambda_i^{(0)})^\alpha, \quad i \geq 1.$$

Obviously, all conditions in Corrollary 3.2 are satisfied for the present situation, hence (1.4) holds. $\square$

To conclude this paper, we present an example where L_0 is the Dirichlet Laplacian on a finite volume domain in $\mathbb{R}^d$, so that L can be taken as high order differential operators on a domain.

Example 3.5. In the situation of Proposition 3.4 but simply take $q_i = i^\theta, i \geq 1$ where $\theta > 0$ is a constant. Let $L_0 := \Delta$ be the Dirichlet Laplace operator on a domain $D \subset \mathbb{R}^d$ with finite volume, and let $\mathbf{m}$ be the normalized volume measure on D. By the Sobolev inequality we have (see [19, Corollaries 1.1 and 3.1])

$$\lambda_i^{(0)} \geq c i^{2/d}, \quad i \geq 1$$

for some $c > 0$. If $\sigma \geq \frac{4}{1+r}$ and

$$\theta > \max\left\{\frac{\sigma + 2\varepsilon - 2}{4(1-\varepsilon)}, \frac{(\sigma + 2\varepsilon - 2)(1-r)}{2\sigma\varepsilon(1+r)}\right\}$$

for some $\varepsilon \in (0,1)$, then (3.7) holds for any $\alpha \in \left(\frac{(2\theta+1)d}{2} \vee \frac{(1-r)d}{2\varepsilon(1+r)}, \frac{\sigma\theta d}{\sigma+2\varepsilon-2}\right]$, so all assertions in Theorems 1.1 and 1.2 hold for the solution to (3.6) according to Proposition 3.4.

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